

HMS for punctured tori and categorical mapping class group actions

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ABSTRACT. Let X_n be a cycle of n projective lines, and \mathbb{T}_n a symplectic torus with n punctures. In this paper we review results of [STZ] and [Si], which establish a version of homological mirror symmetry relating X_n and \mathbb{T}_n , and define on $D^b(\text{Coh}(X_n))$ an action of the pure mapping class group of \mathbb{T}_n .

1. Introduction.

As originally formulated by Kontsevich [K], Homological Mirror Symmetry (from now on, HMS) relates the derived category of coherent sheaves on a smooth projective Calabi-Yau manifold X , $D^b(\text{Coh}(X))$, and the Fukaya category of a compact symplectic manifold \tilde{X} , by stating that if X and \tilde{X} are mirror partners, then $D^b(\text{Coh}(X)) \cong \text{Fuk}(\tilde{X})$. Since its proposal, much work has been done towards establishing Kontsevich's conjecture in important classes of examples, see [PZ, S, Sh], and references therein.

In [STZ], joint with Treumann and Zaslow, we address mirror symmetry in dimension 1, by proving a version of HMS which pairs singular degenerations of elliptic curves, given by cycles of projective lines, and punctured symplectic tori. This result relies on the use of a conjectural model for the Fukaya category of a punctured Riemann surface Σ , which is constructed in terms of a sheaf of dg categories, $\text{CPM}(-)$,¹ defined over the Lagrangian skeleton of Σ .² In [Si], using the theory of spherical objects and twist functors introduced by Seidel and Thomas in [SeT], we test one of the predictions of the mirror symmetry framework developed in [STZ], by showing that the (pure) mapping class group of a punctured torus acts by equivalences on the derived category of a cycle of projective lines.

In this paper we will review these results, by focusing on motivations and examples, and keeping the presentation of the arguments as explicit and concrete as possible. Let X_n be a cycle of projective lines, with n components, and \mathbb{T}_n a symplectic torus with n punctures. We will review the construction of CPM for

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¹CPM stands for ‘constructible plumbing model.’ The name depends on the fact that CPM can be defined in greater generality, and supplies a dg model for the Fukaya category of *plumbings* of cotangent bundles in any dimension [STZ2].

²HMS for punctured spheres has been investigated also in [AAEKO].

\mathbb{T}_n in Section 2. The HMS statement relating the CPM model for $Fuk(\mathbb{T}_n)$ and $\text{Perf}(X_n)$ will be proved in Section 3. We will conclude by giving, in Section 4, a brief overview of the results in [Si], focusing on the case of $D^b(\text{Coh}(X_2))$.

2. A model for the Fukaya category of punctured Riemann surfaces

Starting in 2009, in various talks, Kontsevich has argued [K1] that the Fukaya category of a Stein manifold should have good local-to-global properties, and therefore conjecturally could be recovered as the global sections of a suitable sheaf of dg categories (note also [S1], and [N1]). This is in keeping with previous work of Nadler and Zaslow who, in [NZ] and [N], establish an equivalence between the Fukaya category of exact Lagrangians in a cotangent bundle T^*X , and the dg category of (complexes of cohomologically) constructible sheaves over X , $Sh(X)$.

Following Kontsevich's insight, in [STZ] we equip the Lagrangian skeleton of a punctured Riemann surface Σ with a sheaf of dg categories,³ called $\text{CPM}(-)$, such that its local behavior is dictated by Nadler and Zaslow's work on cotangent bundles, while its global sections are conjecturally quasi-equivalent to the Fukaya category of exact Lagrangians in Σ , $Fuk(\Sigma)$. Before discussing the construction of $\text{CPM}(-)$ in Section 2.2, we collect in Section 2.1 below the necessary background on sheaf theory.

2.1. Microlocal sheaf theory in dimension 1. In [KS], Kashiwara and Schapira explain how to attach to a constructible sheaf $\mathcal{F} \in Sh(X)$ a conical (i.e. invariant under fiberwise dilation) Lagrangian subset of T^*X , called *singular support*, and denoted $SS(\mathcal{F})$. Informally, $SS(\mathcal{F})$ is an invariant encoding the co-directions along which \mathcal{F} does not ‘propagate.’ Rather than giving the exact definition, for which we refer the reader to Section 5.1 of [KS], we will describe in Lemma 2.2 how the singular support works in the cases which will be relevant for us. If $\Lambda \hookrightarrow T^*X$ is a conical Lagrangian subset, denote $Sh(X, \Lambda) \hookrightarrow Sh(X)$ the full subcategory of constructible sheaves $\mathcal{F} \in Sh(X)$ such that $SS(\mathcal{F}) \subset \Lambda$.

Proposition 2.1. *Let X be a 1-dimensional manifold, let $\Lambda \hookrightarrow T^*X$ be a conical Lagrangian subset, and denote $\pi : T^*X \rightarrow X$ the natural projection. The assignment sending a conical open subset $V \subset \Lambda$, to the dg category $Sh(\pi(V), V)$, can be extended to a sheaf of dg categories, denoted $MSh(-)$, over Λ equipped with its natural topology.*

PROOF. The proof of the statement is discussed in Section 3.1 of [STZ], and depends on the microlocal theory of sheaves developed in [KS]. In fact, a similar statement holds in all dimensions [STZ2]. \square

Lemma 2.2. *Let X be a 1-dimensional manifold, and $P = \{p_1, p_2, \dots, p_n\}$ a finite collection of points in X , then $\Lambda_P = X \cup (T_{p_1}^*X \cup \dots \cup T_{p_n}^*X)$ is a conical Lagrangian, and $Sh(X, \Lambda_P) \subset Sh(X)$ coincides with the full subcategory of constructible sheaves which are locally constant on P , and on $X \setminus P$.*

$Sh(X, \Lambda_P)$ admits a very simple combinatorial description in terms of quiver representations. Call \mathcal{S} the partition of X given by the points in P , and by the connected components of $X \setminus P$. Denote $Q_{\mathcal{S}}$ the quiver whose vertices are the

³ $Sh(X)$ is the dg enhancement of the derived category of constructible sheaves of \mathbb{C} -vector spaces over X . From now on, we will refer to objects in $Sh(X)$ simply as ‘constructible sheaves.’ See [KS] for a comprehensive introduction to the subject.

elements of \mathcal{S} , and with an arrow joining $S, S' \in \mathcal{S}$, with that orientation, if and only if S is a point, and S' is a sub-interval such that $S \in \overline{S'}$. For example, if P has cardinality 1, then $Q_{\mathcal{S}}$ is equal to $\bullet \leftarrow \bullet \rightarrow \bullet$ if $X = \mathbb{R}$, and to $\bullet \rightrightarrows \bullet$ if $X = S^1$.

Lemma 2.3. $Sh(X, \Lambda_P) \cong Rep(Q_{\mathcal{S}})$.⁴

PROOF. This statement is part of a larger framework, due to MacPherson, which describes sheaves that are constructible with respect to a given stratification in terms of *exit paths*. If $\mathcal{F} \in Sh(X, \Lambda)$, by taking its stalks over p_1, \dots, p_n , and over points lying on the different components of $X \setminus P$, we obtain a complex of vector spaces for each vertex of $Q_{\mathcal{S}}$. Further, the restrictions maps of \mathcal{F} yield linear maps corresponding to the arrows of $Q_{\mathcal{S}}$. This prescription maps \mathcal{F} to a representation of $Q_{\mathcal{S}}$ in a functorial way, and defines the equivalence. \square

Remark 2.4. Under the assumptions of Lemma 2.3, the sheaf $MSh(-)$ can be described explicitly. Assume, for concreteness, that $X = S^1$, and $P = \{p\}$. Also, fix an orientation on T^*S^1 , and note that this allows us to decompose $T_p^*S^1$ as the union of 0, and two rays, R^+ and R^- , respectively up-ward and down-ward pointing. Below, we describe the sections of $MSh(-)$ on *contractible* open subsets $U \subset \Lambda$, and the assignment defining, on the objects, the restriction functors

$$Res_U : MSh(\Lambda) = Sh(X, \Lambda) \cong Rep(\bullet \rightrightarrows \bullet) \rightarrow MSh(U),$$

the definition on morphisms will be obvious. This is enough to reconstruct $MSh(-)$.

Let $M = V_1 \xrightleftharpoons[g]{f} V_2$ be an object in $Rep(\bullet \rightrightarrows \bullet)$, then

- if $U \subset S^1$, $MSh(U) \cong \mathbb{C} - mod$, and $Res_U(M) = V_2$,
- if $U \subset R^+$, $MSh(U) \cong \mathbb{C} - mod$, and $Res_U(M) = Cone(f)$,
- if $U \subset R^-$, $MSh(U) \cong \mathbb{C} - mod$, and $Res_U(M) = Cone(g)$,
- if $p \in U$, $MSh(U) \cong Rep(\bullet \leftarrow \bullet \rightarrow \bullet)$, and $Res_U(M) = V_2 \xleftarrow{f} V_1 \xrightarrow{g} V_2$.

2.2. The construction of CPM for \mathbb{T}_n . A *ribbon graph* is a graph equipped with a cyclic ordering on the set of half-edges incident to each vertex. Recall that ribbon graphs label cells in the moduli space of punctured Riemann surfaces (see e.g. [P]). Further, if the Riemann surface Σ lies in the cell labelled by Γ_{Σ} , there is an embedding $\Gamma_{\Sigma} \hookrightarrow \Sigma$, and a nicely behaved retraction of Σ onto Γ_{Σ} . In the language of Stein geometry, Γ_{Σ} is the *skeleton* of Σ .

Bracketing issues of valency, given a pair formed by a punctured Riemann surface and its skeleton, $\Gamma_{\Sigma} \hookrightarrow \Sigma$, we can consider an open covering $\{U_i\}_{i \in I}$ of Γ_{Σ} with the property that, for all $i \in I$, there is a symplectomorphism $\phi_i : U_i \rightarrow T^*M_i$, where M_i is a 1-dimensional manifold, and $\Lambda_i := \phi_i(U_i \cap \Gamma_{\Sigma}) \hookrightarrow T^*M_i$ is a conical Lagrangian subset. In conformity with Kontsevich's ansatz, we should be able to recover $Fuk(\Sigma)$ by first applying Nadler and Zaslow's theory [NZ, N] to compute the Fukaya category of each member of the covering family (this yields $Sh(M_i, \Lambda_i)$), and then exploiting the fact that the Fukaya category behaves like a sheaf over Γ_{Σ} .

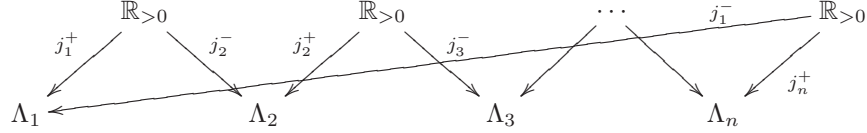
Roughly speaking, the theory developed in [STZ] formalizes this heuristics by constructing a sheaf of dg categories $CPM(-)$ over Γ_{Σ} , which is characterized by the property that, when restricted to $U_i \cap \Gamma_{\Sigma} \cong \Lambda_i$, it coincides with the sheaf $MSh(-)$ over Λ_i introduced in Proposition 2.1. Since $\{U_i\}_{i \in I}$ covers Γ_{Σ} , this prescription

⁴ $Rep(Q_{\mathcal{S}})$ denotes the dg derived category of representations. In the proof below and everywhere in the paper, all functors, such as the stalk functor, are implicitly assumed to be derived.

is actually sufficient to compute sections and restriction functors for $\text{CPM}(-)$ on arbitrary open subsets $U \subset \Gamma_\Sigma$, and therefore determines $\text{CPM}(-)$ uniquely.⁵

Conjecture 2.5. *$\text{CPM}(\Gamma_\Sigma)$ is quasi-equivalent to the Fukaya category of compact exact Lagrangians in Σ , $\text{Fuk}(\Sigma)$*

For the purposes of HMS, it is important to understand how this works for a symplectic torus with n punctures, \mathbb{T}_n . Let Λ_i , $i \in I = \{1, \dots, n\}$, be a collection of n copies of the conical Lagrangian $\Lambda_p := S^1 \cup T_p^* S^1 \hookrightarrow T^* S^1$. Note that, if we fix an orientation on $T^* S^1$, Λ_p acquires a natural structure of ribbon graph. For each $i \in I$ there are open embeddings $j_i^+ : \mathbb{R}_{>0} \cong R_i^+ \hookrightarrow \Lambda_i$, and $j_i^- : \mathbb{R}_{>0} \cong R_i^- \hookrightarrow \Lambda_i$, where R_i^+ and R_i^- are defined as in Remark 2.4. Denote Γ_n the ribbon graph constructed as the push-out of the diagram



Provided that \mathbb{T}_n is equipped with an appropriate complex structure, Γ_n is isomorphic to the skeleton of \mathbb{T}_n .⁶ Also, the Λ_i -s supply an open covering for Γ_n . As $\text{CPM}(-)$ defines a sheaf over Γ_n , its global sections $\text{CPM}(\Gamma_n)$ can be calculated in the usual way, by taking the (homotopy) equalizer of the Čech diagram (2.1) below, and thus, informally, picking out local sections on the Λ_i -s which agree on the overlaps,

$$(2.1) \quad \prod_{i \in I} (\text{CPM}(\Lambda_i) = \text{Sh}(S^1, \Lambda_p)) \xrightleftharpoons[\text{Res}_-]{\text{Res}_+} \prod_{j \in I} (\text{CPM}(\mathbb{R}_{>0}) \cong \mathbb{C} - \text{mod}).$$

Note that the functors Res_+ and Res_- can be explicitly computed, since they are products of restriction functors for $\text{Sh}(S^1, \Lambda_p) \cong \text{Rep}(\bullet \rightrightarrows \bullet)$ which were described in Remark 2.4. We will conclude this section by giving a concrete recipe for constructing homotopy equalizers in dgCat .

Lemma 2.6. *Let $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{D}$ be a diagram in dgCat , and denote \mathcal{E} the dg category having,*

- *as objects, pairs (C, u) , where $C \in \mathcal{C}$, and u is a degree zero, closed morphism $u : F(C) \cong G(C)$, which becomes invertible in the homotopy category,*
- *as morphisms, pairs $(f, H) \in \text{hom}^k(C, C') \oplus \text{hom}^{k-1}(F(C), G(C'))$, with differential given by $d(f, H) = (df, dH - (u'F(f) - G(f)u))$. The composition is obvious.*

Then \mathcal{E} , endowed with the natural forgetful functor $\mathcal{E} \rightarrow \mathcal{C}$, is a homotopy equalizer for F and G .

PROOF. Lemma 2.6 depends on the availability of an explicit construction of the path object $P(\mathcal{D})$ for \mathcal{D} , which can be found in Lemma 4.1 of [Tab1]. This allows us

⁵This informal account disregards various technical aspects of the theory, for which see [STZ].

⁶Note that, although a different choice of complex structure on \mathbb{T}_n could alter the geometry of the skeleton, this would not affect, up to quasi-equivalence, the global sections of $\text{CPM}(-)$.

to compute the homotopy equalizer in the usual way, by taking appropriate fibrant replacements. We leave the details to the reader. \square

3. CPM and mirror symmetry for degenerate elliptic curves

Let X_n be a cycle of n projective lines. That is, X_n is a connected reduced curve with n nodal singularities, such that its normalization $\tilde{X}_n \xrightarrow{p} X_n$ is a disjoint union of n projective lines D_1, \dots, D_n , with the property that the pre-image along π of the singular set intersects each D_i in exactly two points. Theorem 3.3 below yields a description of $\mathcal{P}\text{erf}(X_n)$ as a suitable homotopy equalizer of dg categories. This is one of the key steps in the proof of HMS for X_n , which will be presented in Section 3.2.

3.1. Perfect complexes over a nodal curve. It will be convenient to make use of the following general result, according to which $\mathcal{P}\text{erf}(-)$ defines a sheaf of dg categories for the Zariski topology.

Theorem 3.1 (see [To], Proposition 11). *Let $X = U \cup V$, where U and V are two Zariski open subschemes. Then the following square:*

$$\begin{array}{ccc} \mathcal{P}\text{erf}(X) & \longrightarrow & \mathcal{P}\text{erf}(U) \\ \downarrow & & \downarrow \\ \mathcal{P}\text{erf}(V) & \longrightarrow & \mathcal{P}\text{erf}(U \cap V) \end{array}$$

is a fiber product of dg categories.

Lemma 3.2. *Let X be a nodal curve, with normalization $\tilde{X} \xrightarrow{\pi} X$, then for every $\tilde{\mathcal{F}}$ vector bundle on \tilde{X} and isomorphism $u : \sigma^*(\tilde{\mathcal{F}}) \rightarrow \tau^*(\tilde{\mathcal{F}})$, the assignment:*

$$U \subset^{\text{open}} X \mapsto \{s \in \tilde{\mathcal{F}}(\pi^{-1}(U)) \mid u(\sigma^*(s)) = \tau^*(s)\},$$

defines a vector bundle $\tilde{\mathcal{F}}^u$ on X such that $\pi^(\tilde{\mathcal{F}}^u) \cong \tilde{\mathcal{F}}$. Conversely, if \mathcal{F} is a vector bundle on X such that $\pi^*\mathcal{F} \cong \tilde{\mathcal{F}}$, then $\mathcal{F} \cong \tilde{\mathcal{F}}^u$ for some isomorphism $u : \sigma^*(\tilde{\mathcal{F}}) \rightarrow \tau^*(\tilde{\mathcal{F}})$.*

PROOF. See Proposition 4.4 in [L]. \square

Recall that a nodal curve is a curve having only double points as singularities.

Theorem 3.3. *Let X be a nodal curve, with singular set Z , and normalization $\pi : \tilde{X} \rightarrow X$. Let $\sigma, \tau : Z \rightarrow X$ be two non-overlapping sections of $\pi^{-1}(Z) \rightarrow Z$, then the diagram*

$$\mathcal{P}\text{erf}(X) \xrightarrow{\pi^*} D^b(\text{Coh}(\tilde{X})) \xrightleftharpoons[\tau^*]{\sigma^*} D^b(\text{Coh}(Z))$$

is an equalizer of dg categories.

PROOF. Since limits commute with limits, it is sufficient, after Theorem 3.1, to prove the claim for affine X , so we will restrict to this case. Let E be the equalizer of the diagram

$$D^b(\text{Coh}(\tilde{X})) \xrightleftharpoons[\tau^*]{\sigma^*} D^b(\text{Coh}(Z))$$

constructed according to the prescriptions of Lemma 2.6. Recall that the objects of E are pairs $(\tilde{\mathcal{F}}, u)$, where $\tilde{\mathcal{F}}$ is an object of $D^b(\text{Coh}(\tilde{X}))$, and u is a degree zero, closed morphism $u : \sigma^* \tilde{\mathcal{F}} \cong \tau^* \tilde{\mathcal{F}}$, which becomes invertible in the homotopy category. The morphisms of E are pairs $(f, H) \in \text{hom}^k(\tilde{\mathcal{F}}, \tilde{\mathcal{G}}) \oplus \text{hom}^{k-1}(\sigma^* \pi^* \tilde{\mathcal{F}}, \tau^* \pi^* \tilde{\mathcal{G}})$, and the differential is given by $d(f, H) = (df, dH - (u' \sigma^*(f) - \tau^*(f)u))$.

Fix a natural equivalence $\alpha : \sigma^* \pi^* \cong \tau^* \pi^*$. As $\text{Perf}(X)$ is generated by line bundles, and E is generated by objects of the form $(\tilde{\mathcal{F}}, u)$ with $\tilde{\mathcal{F}}$ a line bundle on \tilde{X} , it is sufficient to define a (quasi-)equivalence ψ between these two linear sub-categories. Define ψ as follows,

- if \mathcal{F} is a line bundle on X , then $\psi(\mathcal{F}) = (\pi^* \mathcal{F}, \sigma^* \pi^* \mathcal{F} \xrightarrow{\alpha} \tau^* \pi^* \mathcal{F})$,
- if \mathcal{F}, \mathcal{G} are line bundles on X , and $f \in \text{hom}^k(\mathcal{F}, \mathcal{G})$, then $\psi(f) = (\pi^* f, 0)$.

Consider a line bundle $\tilde{\mathcal{F}}$ over \tilde{X} . It follows from Lemma 3.2 that the set of isomorphism classes of line bundles \mathcal{F} on X such that $\pi^* \mathcal{F} \cong \tilde{\mathcal{F}}$ carries a transitive action by $(\mathbb{C}^*)^{|Z|}$ (given by pointwise rescaling the ‘compatibility’ isomorphisms u , see Lemma 3.2). Further, the same is true for the set of isomorphism classes of objects of $(\tilde{\mathcal{G}}, v) \in E$, such that $(\tilde{\mathcal{G}}, v) \cong (\tilde{\mathcal{F}}, u)$ for some $u \in \text{hom}^0(\sigma^* \tilde{\mathcal{F}}, \tau^* \tilde{\mathcal{F}})$. Essential surjectivity follows from the fact that ψ defines a $(\mathbb{C}^*)^{|Z|}$ -equivariant map between these two sets of isomorphism classes.

We shall prove next that ψ is quasi-fully faithful, i.e. that the map between hom-complexes defined by ψ induces an isomorphism in the homotopy category. Denote $\text{Ho}E$ the homotopy category of E . It is sufficient to show that for all line bundles \mathcal{F} on X , and for all $i \in \mathbb{N}$,

$$\psi : \text{Hom}_X^i(\mathcal{O}_X, \mathcal{F}) (= H_X^i(\mathcal{F})) \xrightarrow{\cong} \text{Hom}_{\text{Ho}E}^i(\psi(\mathcal{O}_X), \psi(\mathcal{F})).$$

Note that, as X and \tilde{X} are affine, cohomology vanishes in positive degree. It follows that $\text{Hom}_{\text{Ho}E}^i(\psi(\mathcal{O}_X), \psi(\mathcal{F})) = 0$ for all $i > 0$.⁷ Further, in degree-zero, the hom-space fits in the following short exact sequence

$$0 \rightarrow \text{Hom}_{\text{Ho}E}^0(\psi(\mathcal{O}_X), \psi(\mathcal{F})) \rightarrow \text{Hom}_{\tilde{X}}^0(\mathcal{O}_{\tilde{X}}, \pi^* \mathcal{F}) \rightarrow \text{Hom}_{\tilde{Z}}^0(\sigma^* \mathcal{O}_{\tilde{X}}, \tau^* \pi^* \mathcal{F}) \rightarrow 0.$$

Thus, proving fully faithfulness boils down to showing exactness of

$$(3.1) \quad 0 \rightarrow H_X^0(\mathcal{F}) \xrightarrow{\pi^*} H_{\tilde{X}}^0(\pi^* \mathcal{F}) \rightarrow H_{\tilde{Z}}^0(\tau^* \pi^* \mathcal{F}) \rightarrow 0.$$

Now, (3.1) is obtained by taking global sections of the sequence

$$0 \rightarrow \mathcal{F} \rightarrow \pi_*(\pi^* \mathcal{F}) \rightarrow \pi_* \tau_* \tau^*(\pi^* \mathcal{F}) \rightarrow 0,$$

which is exact (see the proof of Proposition 4.4 of [L]). Since X is affine, taking global section is an exact operation, and this concludes the proof of Theorem 3.3. \square

3.2. HMS for nodal elliptic curves. In this section we will prove that the category of perfect complexes over X_n is quasi-equivalent to $\text{CPM}(\Gamma_n)$. Granting Conjecture 2.5, this result confirms well known mirror symmetry heuristics, which suggest that the mirror of X_n should be a symplectic torus with n punctures, \mathbb{T}_n .⁸

Theorem 3.4 ([Be]). *There is an equivalence $\beta : D^b(\text{Coh}(\mathbb{P}^1)) \xrightarrow{\cong} \text{Rep}(\bullet \rightrightarrows \bullet)$.*

⁷Note that $\text{Hom}_{\text{Ho}E}^1(\psi(\mathcal{O}_X), \psi(\mathcal{F}))$ vanishes, since it is isomorphic to the quotient of $\text{Hom}_{\tilde{Z}}^0(\sigma^* \mathcal{O}_{\tilde{X}}, \tau^* \pi^* \mathcal{F}) \cong \mathbb{C}$ by the image of the differential, which is easily seen to be surjective.

⁸Kontsevich announced related results in [K1]. HMS for the nodal \mathbb{P}^1 is also treated in [LP].

PROOF. For the proof, see Beilinson's famous paper [Be], which provides analogous descriptions of $D^b(\text{Coh}(\mathbb{P}^n))$ for any n . The functor β can be defined as follows. Fix a basis x_0, x_1 for $H^0(\mathcal{O}(1))$, and set

$$\mathcal{F} \in D^b(\text{Coh}(\mathbb{P}^1)) \mapsto \beta(\mathcal{F}) = R\Gamma(\mathcal{F} \otimes \mathcal{O}(-1)) \begin{matrix} \xrightarrow{\cdot x_0} \\ \xrightarrow{\cdot x_1} \end{matrix} R\Gamma(\mathcal{F}) \in \text{Rep}(\bullet \rightrightarrows \bullet),$$

with the obvious definition on morphisms. \square

In view of results of Nadler and Zaslow [NZ, N] discussed in Section 2.2, Proposition 3.4, combined with the equivalence $\text{Rep}(\bullet \rightrightarrows \bullet) \cong \text{Sh}(S^1, \Lambda_p)$, yields a homological mirror symmetry statement pairing $D^b(\text{Coh}(\mathbb{P}^1))$, and a suitable Fukaya category of exact Lagrangians in T^*S^1 . This was explained as an instance of T-duality by Fang [F], and fits in the framework of the *coherent-constructible correspondence* developed by Fang, Liu, Treumann and Zaslow (see [FLTZ]), which is one of the starting points for the project of [STZ].⁹

Theorem 3.5 ([STZ]). *Let X_n be a cycle of n projective lines. There is a quasi-equivalence $\phi : \text{Perf}(X_n) \cong \text{CPM}(\Gamma_n)$.*¹⁰

PROOF. Let $Z \hookrightarrow X_n$ be the singular set. Pick two non-overlapping sections σ , and τ of $p^{-1}(Z) \rightarrow Z$, as in Theorem 3.3, and choose an identification $D^b(\text{Coh}(Z)) \cong \prod_{j=1}^{j=n} \mathbb{C} - \text{mod}$. The proof is encoded in the following diagram,

$$\begin{array}{ccccc} \text{Perf}(X_n) & \longrightarrow & \prod_{i=1}^{i=n} D^b(\text{Coh}(\mathbb{P}^1)) & \begin{matrix} \xrightarrow{\sigma^*} \\ \xrightarrow{\tau^*} \end{matrix} & \prod_{j=1}^{j=n} \mathbb{C} - \text{mod} \\ \downarrow \phi & & \downarrow \beta & & \downarrow \rho \\ \text{CPM}(\Gamma_n) & \longrightarrow & \prod_{i=1}^{i=n} \text{Rep}(\bullet \rightrightarrows \bullet) & \begin{matrix} \xrightarrow{\text{Res}^+} \\ \xrightarrow{\text{Res}^-} \end{matrix} & \prod_{j=1}^{j=n} \mathbb{C} - \text{mod}. \end{array}$$

In fact, we can choose ρ in such a way that $\rho \circ \sigma^* \cong \text{Res}^+ \circ \beta$, and $\rho \circ \tau^* \cong \text{Res}^- \circ \beta$, where, abusing notation, we are denoting $\times_{i=1}^{i=n} \beta$ simply by β . This implies that the equalizer of σ^*, τ^* is quasi-equivalent to the equalizer of $\text{Res}^+, \text{Res}^-$ (see diagram 2.1), and yields $\phi : \text{Perf}(X_n) \xrightarrow{\cong} \text{CPM}(\Gamma_n)$. \square

4. A mapping class group action on $D^b(\text{Coh}(X_n))$

Since the group of symplectic automorphisms of \mathbb{T}_n acts by auto-equivalences on $\text{Fuk}(\mathbb{T}_n)$, HMS predicts the existence of a mirror action on $D^b(\text{Coh}(X_n))$.¹¹ This is the content of the main theorem of [Si], which we state below.

Theorem 4.1 ([Si]). *Let $\text{PM}(\mathbb{T}_n)$ be the pure mapping class group of \mathbb{T}_n , then, up to shift, there is an action of $\text{PM}(\mathbb{T}_n)$ over $D^b(\text{Coh}(X_n))$.*

Recall that the mapping class group of an oriented surface Σ , $\text{MCG}(\Sigma)$, is the group of symplectic automorphisms of Σ , up to isotopy. The pure mapping class group is the subgroup $\text{PM}(\Sigma) \hookrightarrow \text{MCG}(\Sigma)$ generated by Dehn twists (see

⁹The significance for mirror symmetry of the equivalence $D^b(\text{Coh}(\mathbb{P}^1)) \cong \text{Sh}(S^1, \Lambda_p)$ was first advocated by Bondal [B], in the context of HMS for weighted projective spaces.

¹⁰The result proved in [STZ] is actually more general, and extends to appropriate *stacky* degenerations of elliptic curves.

¹¹Note in fact that, although the HMS statement of Section 3.2 involves $\text{Perf}(X_n)$, it is possible to show that $D^b(\text{Coh}(X_n))$ and $\text{Perf}(X_n)$ have the same group of auto-equivalences.

[FM]). Theorem 4.1 generalizes previous work of Seidel and Thomas [SeT], and Burban and Kreussler [BK], who established, respectively, the existence of an $SL(2, \mathbb{Z})$ -action over $D^b(\text{Coh}(X))$, where X is a smooth elliptic curve, and over $D^b(\text{Coh}(X_1))$.¹² The proof of Theorem 4.1 depends on the availability of an explicit presentation of $\text{PM}(\mathbb{T}_n)$, worked out in the first Section of [Si], and on Seidel and Thomas' theory of *spherical objects* and *twist functors* [SeT], which is designed precisely to test this aspect of mirror symmetry.

In the rest of this section, we shall briefly describe the proof of Theorem 4.1 for $n = 2$, leaving out most details, and referring the reader to [Si] for the general case. A presentation of $\text{PM}(\mathbb{T}_2)$ can be found in [PS], and is reproduced below.

Proposition 4.2. *The pure mapping class group $\text{PM}(\mathbb{T}_2)$ is generated by T_α , T_{β_1} and T_{β_2} ,¹³ with relations*

- (Braid relations) $T_{\beta_i} T_{\beta_j} = T_{\beta_j} T_{\beta_i}$, $T_{\beta_i} T_\alpha T_{\beta_i} = T_\alpha T_{\beta_i} T_\alpha$,
- (G-relation) $(T_{\beta_1} T_\alpha T_{\beta_2})^4 = 1$.

Following the discussion in Section 1 of [SeT], the group acting on $D^b \text{Coh}(X_2)$ is going to be a suitable central extension of $\text{PM}(\mathbb{T}_2)$, whose elements should be viewed as *graded* symplectic automorphisms of the mirror of X_2 , i.e. \mathbb{T}_2 .

Definition 4.3. Define $\widetilde{\text{PM}}(\mathbb{T}_2)$ as the \mathbb{Z} -central extension of $\text{PM}(\mathbb{T}_2)$, generated by T_α , T_{β_i} , $i = 1, 2$, and a central element t subject to the following relations

- (Braid relations), as in Proposition 4.2
- (\tilde{G} -relation) $(T_{\beta_1} T_\alpha T_{\beta_2})^4 = t^2$.

Let x_1 and x_2 be two smooth points lying on different components of X_2 , then

Theorem 4.4. *The assignment*

- for all $i = 1, 2$, $T_{\beta_i} \mapsto T_{\kappa(x_i)}$,
- $T_\alpha \mapsto T_{\mathcal{O}}$, and
- $t \mapsto [1]$,

defines an action of $\widetilde{\text{PM}}(\mathbb{T}_2)$ on $D^b(\text{Coh}(X_2))$.

Note that the assignment described in Theorem 4.4 is compatible with mirror symmetry considerations, according to which \mathcal{O} and $\kappa(x_i)$ should be mirror to Lagrangian branes whose supports are isotopic, respectively, to α and β_i . We state below two lemmas, formulated for a general cycle of projective lines X_n , which will be important for proving Theorem 4.4.

Lemma 4.5. *Let $F : D^b(\text{Coh}(X_n)) \rightarrow D^b(\text{Coh}(X_n))$ be an auto-equivalence of triangulated categories. If*

- $F(\mathcal{O}) \cong \mathcal{O}$, and
- for all $i \in \{1 \dots n\}$, $F(\kappa(x_i)) \cong \kappa(x_i)$,

then there exists an isomorphism $f : X_n \rightarrow X_n$, such that F is naturally equivalent to $f^ : D^b(\text{Coh}(X_n)) \rightarrow D^b(\text{Coh}(X_n))$.*

¹²Recall that the mapping class groups of the torus and of the once punctured torus are both isomorphic to $SL(2, \mathbb{Z})$.

¹³The generators T_α, T_{β_i} are given by isotopy classes of Dehn twists along simple closed curves $\alpha, \beta_i \hookrightarrow \mathbb{T}_2$. Explicit representatives can be described as follows. Identify the torus \mathbb{T} with $[0, 1] \times [0, 1] / \sim$, and set $\mathbb{T}_2 = \mathbb{T} \setminus \{p_1 = (0, 0), p_2 = (\frac{1}{2}, 0)\}$, then $\alpha = [0, 1] \times \{\frac{1}{2}\}$, and $\beta_i = \{\frac{i}{3}\} \times [0, 1]$, $i = 1, 2$.

PROOF. See Lemma 3.3 of [Si]. The key fact is that if F preserves \mathcal{O} and $\kappa(x_i)$, then it will induce an automorphism of the homogeneous coordinate ring associated to the ample line bundle $\mathcal{L} = \mathcal{O}(x_1 + x_2 + \dots + x_n)$. This implies the existence of an isomorphism $f : X_n \rightarrow X_n$ such that $F = f^*$, when restricted to the linear sub-category having as objects the tensor powers of \mathcal{L} . The claim then follows as in the proof of Theorem 3.1 of [BO]. Note that, under the assumptions of the theorem, if $n > 2$, f has to be the identity. For $n = 2$, f might be non-trivial, but has to be involutive, i.e. $f^2 = id$. \square

Lemma 4.6. *Let $x \in X_n$ be a smooth point, then*

- $T_{\kappa(x)} \cong - \otimes \mathcal{O}(x)$,
- $T_{\mathcal{O}}(\kappa(x)) \cong \mathcal{O}(-x)[1]$,
- $T_{\mathcal{O}}(\mathcal{O}(x)) \cong \kappa(x)$,
- $T_{\mathcal{O}}(\mathcal{O}) \cong \mathcal{O}$.

PROOF. The first isomorphism is proved in [SeT], Section 3.d. For the other isomorphisms, see Lemma 2.13 in [BK]. \square

PROOF OF THEOREM 4.4. The braid relations follow from Proposition 2.13 of [SeT]. It remains to check that $T_{\mathcal{O}}, T_{\kappa(x_i)}, i = 1, 2$ satisfy the \tilde{G} -relation. Simply by keeping track of the isomorphisms collected in Lemma 4.6, and applying the braid relations, one can see that

- $(T_{\kappa(x_1)} T_{\mathcal{O}} T_{\kappa(x_2)})^2(\mathcal{O}) \cong \mathcal{O}[1]$, and
- $(T_{\kappa(x_1)} T_{\mathcal{O}} T_{\kappa(x_2)})^2(\kappa(x_1)) \cong \kappa(x_2)[1]$, $(T_{\kappa(x_1)} T_{\mathcal{O}} T_{\kappa(x_2)})^2(\kappa(x_2)) \cong \kappa(x_1)[1]$.

Let's check this for $\kappa(x_1)$:

$$(T_{\kappa(x_1)} T_{\mathcal{O}} T_{\kappa(x_2)})(T_{\kappa(x_1)} T_{\mathcal{O}} T_{\kappa(x_2)})(\kappa(x_1)) \cong (T_{\kappa(x_1)} T_{\mathcal{O}} T_{\kappa(x_2)})(\mathcal{O}[1]) \cong \kappa(x_2)[1].$$

Consider an involution $\sigma : X_2 \rightarrow X_2$ such that $\sigma(x_1) = x_2$, and $\sigma(x_2) = x_1$. It follows from Lemma 4.5, and the comments made at the end of its proof, that there is an involution $f : X_2 \rightarrow X_2$, and a natural equivalence $(T_{\kappa(x_1)} T_{\mathcal{O}} T_{\kappa(x_2)})^2 \cong f^* \sigma^*[1]$. As σ and f commute, by taking the square of this natural equivalence, one gets

$$(T_{\kappa(x_1)} T_{\mathcal{O}} T_{\kappa(x_2)})^4 \cong (f^* \sigma^*[1])(f^* \sigma^*[1]) \cong (f^*)^2 (\sigma^*)^2 [2] \cong [2].$$

This concludes the proof of the theorem. \square

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